Singular Vector Perturbations without Linear or Adjoint Models

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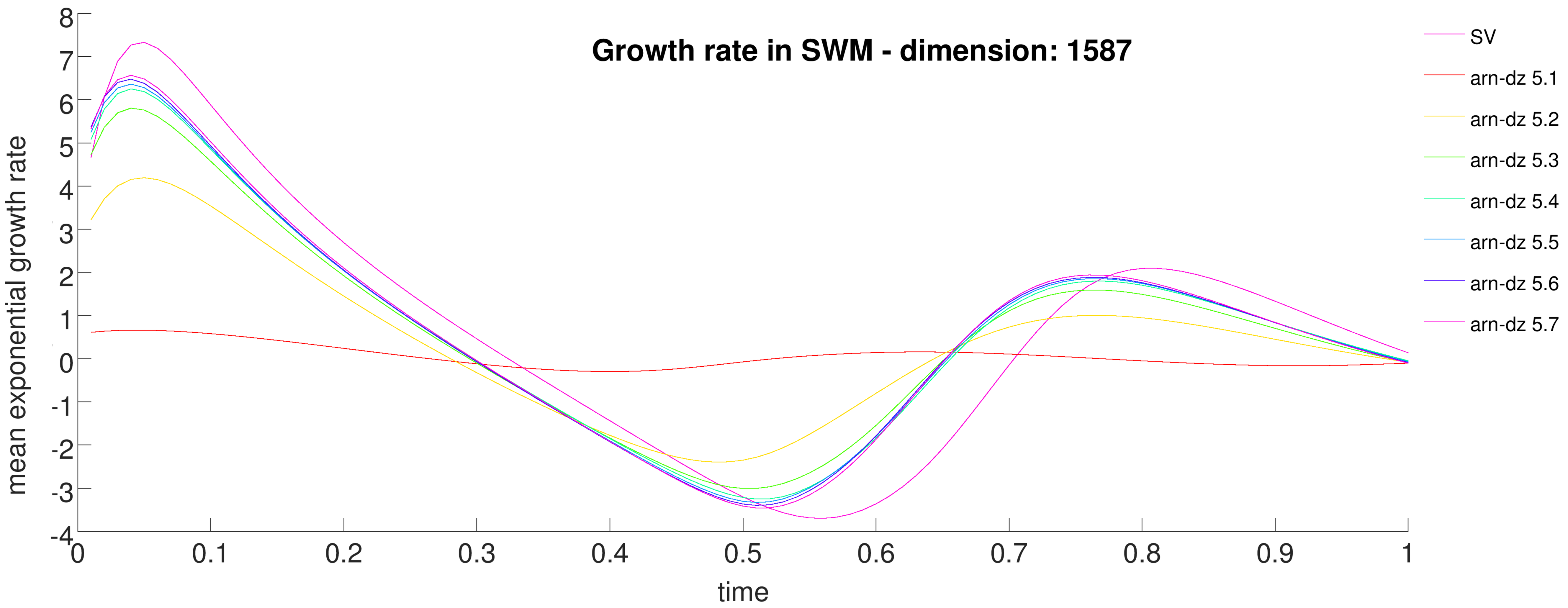

Figure 1: *Mean exponential growth rate of SV and Arnoldi generated perturbations with several iterations, started with five inital vectors each. Results for up to seven iterations.*

In weather forecasting ensemble prediction systems (EPS) are widely used to estimate forecast uncertainty. Forecast errors may arise from uncertainties in initial conditions and it is of great interest to identify the subspace of growing perturbations at initial time. These modes can be obtained by using suitable singular vector (SV) perturbations. Unfortunately, calculating SVs is very expensive, especially in high resolution systems. We present an efficient method for approximating SVs from forecasts with the full non-linear model.

Basic idea

The approximation method is based on three parts:

- Definition of a suitable propagator matrix
- Construction of a proper subspace
- Computation of the leading SVs and mapping to the original space

Propagator

$$Y := \left(\frac{\partial}{\partial x_1}\varphi_T(x_0), \quad .. \quad, \frac{\partial}{\partial x_n}\varphi_T(x_0)\right)$$

$Y \in \mathbb{R}^{n\times n}$

$n \in \mathbb{N}_+$ - dimension of the system

$x_0 \in \mathbb{R}^n$ - initial state

$x_i \in \mathbb{R}$ - i-th component of x_0

$T \in \mathbb{R}_+$ - time intervall

$\varphi_T(x_0) \in \mathbb{R}^n$ - developed state

Approx. of matrix-vector products

$$Yv \approx \frac{1}{h}(\varphi_T(x_0 + hv) - \varphi_T(x_0))$$

$v \in \mathbb{R}^n$ - arbitrary unit vector

Arnoldi - Matrix dimensions

$$\left(\; Y \; \right)\left(Q_m\right) \approx \left(Q_m\right)\left(H_m\right)$$

Figure 2: Schematic representation of the relationship and dimensional size between the Arnoldi matrices and the full matrix *Y*.
The Krylov subspace can be much smaller than the original system.

Block-Arnoldi Approximation

For a given Matrix Y the Krylov subspace method of Arnoldi can be used to obtain an approximation of an invariant subspace H_m and also an orthonormal basis Q_m thereof. We use a block version of Arnoldi iteration, which allows to start with more than one initial vector [1].

$$Y\,Q_m \;=\; Q_m\;H_m\; + \text{“residuum”}$$

$H_m \in \mathbb{R}^{ml\times ml}$ – approximation of Y in Krylov subspace

$Q_m \in \mathbb{R}^{n\times ml}$ – orthonormal basis

l – number of inital vectors

m – number of iteration loops

Therefore (block) Arnoldi iteration needs just (the approximation of) matrix-vector products, but not the knowledge of the whole matrix Y. Hence, this method is matrix-free.

If the residuum vanishes, one can show that exact SVs of Y can be computed directly, by using the SVD of H_m. But in reality, one will just be able to *approximate* an invariant subspace. However, it is realistic to expect that we can get a reasonable approximation to the leading SVs, even in relative small subspaces. Consequently, this should lead to strong growing perturbations, which can be computed efficiently.

Results

Numerical tests are done with the hyperbolic basic shallow water model (SWM), solved on a two-dimensional domain. A detailed description of that model can be found in, e.g., [2].
The used discretisized model has 1587 degrees of freedom. Numerical solutions are computed with the Lax-Wendroff scheme [3],[4].

Fig. 1 shows the development of the mean logarithmized perturbation growth rate (*mean exponential growth rate*) relative to the reference trajectory. The mean is obtained from 100 perturbations, which are placed at randomly chosen points of the reference trajectory. The optimization time is set to $\mathrm{T} = 0.2$.

Arnoldi perturbations growth

l \ m	1	2	3	4	5	6	7
1	0	6.7	11	14	16	18	19
2	4.6	25	54	60	64	66	67
3	6.8	44	70	74	77	78	79
4	8.9	54	76	81	83	84	85
5	9.8	59	79	85	86	87	88

Table 1: *Integral of the mean exponential growth curves over the intervall [0,T], of Arnoldi generated perturbations. The given values are in percentage of the correspondent SV growth.*

Relative Computation time - SWM

l \ m	1	2	3	4	5	6	7
1	0.1	0.1	0.2	0.2	0.3	0.4	0.4
2	0.1	0.2	0.4	0.5	0.6	0.7	0.9
3	0.2	0.4	0.6	0.7	0.9	1.1	1.3
4	0.3	0.5	0.7	1.0	1.2	1.5	1.7
5	0.3	0.6	0.9	1.2	1.6	1.9	2.2

Table 2: *Computation time within a two-dimensional SWM. The time is given in percent of the full computation of Y and its SV's.*

Besides the optimal SV perturbation strategy, Fig. 1 shows results of Arnoldi generated perturbations with five initial vectors each. The different curves show the performance of up to seven iteration loops. Hence, the perturbations are computed in up to 35-dimensional subspaces.

The choice of the initial vectors for Arnoldi is quite important. Here, differences between two nearby states of the past trajectory are taken to start the Algorithm.

Table 1 gives an overview of the total growth obtained by Arnoldi perturbations with several numbers of initial vectors and iterations.

The computational costs are given in a similar way in Table 2. In both cases the measurements are given in percent of the corresponding results of the leading SV.

Conclusion

We used a matrix-free Arnoldi method for approximating SVs in Krylov subspaces. It avoids calculating the tangent linear operator explicitly and can be used to optimize any initial set of perturbations with respect to error growth.
Our intention is to use this algorithm for improving the match of predicted patterns of ensemble spread and observed forecast errors, especially at the beginning of the forecast.

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[3] B. Wendroff P. Lax, *Systems of conservation laws*, Com. Pure Appl. Math **13** (1960), 217-237

[4] S.K. Ray M. Saiduzzaman, *Comparison of Numerical Schemes for Shallow Water Equation*, Global Journal of Science Frontier Research **13** (2013)

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