Data Assimilation for a New Stochastic Shallow Water Model

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Introduction

We investigate a data assimilation problem consisting of a two-dimensional signal - the **Stochastic Rotating Shallow Water (SRSW) model** - and an observation process corresponding to pointwise measurements of the pressure field between 30 and 60 degrees north latitude. **The SRSW model successfully addresses the unresolved processes issue**. It models the motion of a compressible shallow fluid below a free surface - a type of fluid that has close analogues to the atmosphere. In the SRSW model, the evolution of a two-dimensional rotating system is represented by a system of stochastic partial differential equations. The deterministic part of the SPDEs consists of a classical rotating shallow water system, while the stochastic part involves a transport-type noise. The resulting stochastic system preserves important physical properties of the original deterministic equations.

Stochasticity into Shallow Water Equations

Deterministic - velocity form:

$$d(\epsilon u + R) - udt \times curl(\epsilon u + R) + \nabla\left(\frac{\epsilon}{2}|u|^2 + k\right)dt = 0$$

Stochastic - velocity form[4]:

$$\epsilon du - dy_t \times curl(\epsilon u + R) + \nabla\left(\sum_i \xi_i \circ dW_t^i \cdot (\epsilon u + R)\right) = \nabla\left(-\frac{\epsilon}{2}|u|^2 - k\right)dt$$

where ϵ is the Rossby number, u is the horizontal fluid velocity vector, h is the thickness of the fluid, $\omega = \hat{z} \cdot curl\,(\epsilon u + R)$ is total vorticity, $\hat{z}$ is a unit vector pointing away from the centre of the Earth, R is the vector potential of the zero divergence rotation rate about the vertical direction, $k := \frac{h}{\epsilon \mathcal{F}}$ and $\mathcal{F}$ is the Froude number.

Motivation

The weather and climate system is mostly represented by large-scale patterns, but previous studies have shown that specific small-scale physical mechanisms have a strong impact on the large-scale phenomena. The **introduction of stochasticity into ideal fluid dynamics** provides highly efficient tools when trying to mimic the small-scale physical processes which generally remain unresolved in a purely deterministic framework. Our aim is to investigate the well-posedness of a mathematical system dealing with these issues and to implement it in real-world applications. This will lead to more precision in weather prediction techniques and climate change issues.

The ξ_i vector fields

In the SRSW model the velocity vector field is perturbed by a transport-type noise and vorticity is transported along the stochastically perturbed trajectory[4]

$$dy_t = udt + \sum_i \xi_i \circ dW_t^i,$$

where W_t^i are independent Brownian motions.

The ξ_i vector fields are divergence-free, time-independent, derived from the underlying physics and they correspond to spatial correlations defined by a velocity-velocity correlation matrix. These parameters can be estimated by comparing the fine grid and the coarse grid Lagrangian trajectories. It has been proven recently in [1] that for an incompressible fluid this spatial structure can be estimated from data in such a way that an ensemble of this type of stochastic paths will successfully track the large-scale behaviour of the original deterministic system.

Data Assimilation for the Stochastic Rotating Shallow Water Model

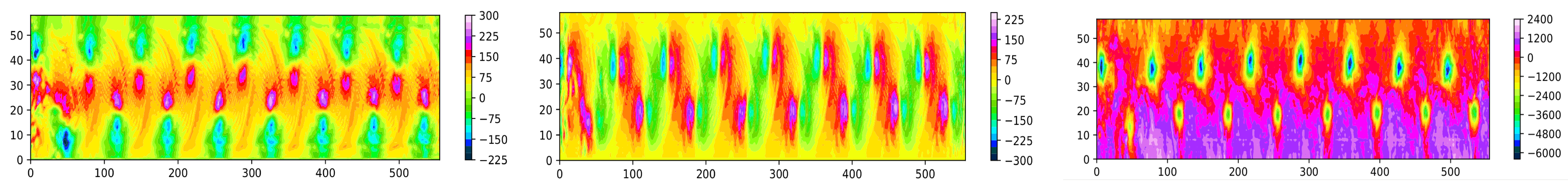
Our research is intended to provide an analysis of a data assimilation problem where the **signal** satisfies the following SPDE:

$$d(\epsilon u + R) - (u \times curl(\epsilon u + R))dt + \sum_i \Big((\xi_i \cdot \nabla)(\epsilon u + R) + (\epsilon u + R)\cdot \nabla \xi_i\Big) \circ dW_t^i = \nabla\left(-\frac{\epsilon}{2}|u|^2 - k\right)dt$$

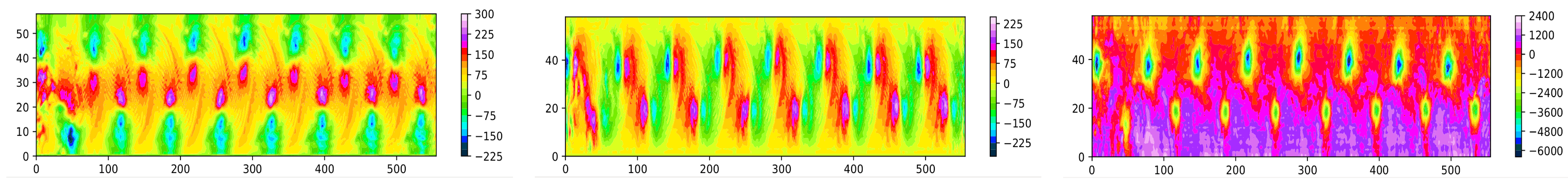
$$dh + \nabla \cdot (hu)dt + \sum_i (\nabla \cdot (\xi_i h)) \circ dW_t^i = 0$$

$$dq + (u \cdot \nabla q)dt + \sum_i (\xi_i \cdot \nabla q) \circ dW_t^i = 0$$

where $q = \omega / h$ represents potential vorticity. Mathematical well-posedness of the vorticity equation in the 3D case has been proven in [2]. The well-posedness for the 2D case is now being prepared in [3] using different techniques. The observations are pointwise measurements corresponding to the pressure field between 30 and 60 degrees north latitude, collected using commercial aircraft (DWD). In the first phase we use an ideal simulation of the *truth*, when the model is run at fine resolution for 500 time steps. The solution has the following form (zonal velocity, meridional velocity, pressure):



The data assimilation process is then performed for 500 ensemble members, 200 time steps, using 5 observations and a standard particle filter:



Code by P. J. van Leeuwen

Conclusion

The stochastic model studied here has been derived in [4] by means of stochastic variational principles, and **it preserves the physical properties** of the real system. We are currently developing an extensive analysis concerning both the mathematical well-posedness and the numerical reliability of the SRSW model. **Positive results have already been achieved (see [3] and above)**. As a result, **the models used in data assimilation could be improved significantly**, leading to more precision in weather prediction and climate change issues.

References

[1] C., Cotter et. al. *Numerically Modelling Stochastic Lie Transport in Fluid Dynamics*, 2018.
[2] D. Crisan, F. Flandoli, D, Holm, *Solution properties of a 3D stochastic Euler fluid equation*, 2017.
[3] D. Crisan, O. Lang, *Global well-posedness of a 2D stochastic Euler equation with Lie transport type noise* (in preparation)
[4] D. Holm, *Variational Principles for Stochastic Fluid Dynamics*, 2015.
[5] G. Nakamura, R. Potthast, *Inverse Modeling - An introduction to the theory and methods of inverse problems and data assimilation, 2015.*
[6] P. J. van Leeuwen, *Aspects of Particle Filtering in High-Dimensional Spaces*, 2015.