Mixed Gaussian-Lognormal based Variational Data Assimilation

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Plan of talk

- Introduction to the mixed Gaussian-lognormal Distribution
- Properties of the mixed distribution
- Plots of the mixed distributions
- Applying the mixed distribution to a variational formulation
- Application of the mixed distribution to microwave brightness temperature based temperature-humidity retrievals
- Comparison with Gaussian and the logarithmic transform approach.
- Lognormal based quality control measures.

Mixed Gaussian-Lognormal distribution

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The mixed distribution in its bivariate formulation is defined by

$$MX(\mu_G, \mu_L, \sigma_G, \sigma_L, \rho_{mx}) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}_{mx}| 2\pi x_2}} \exp\left\{-\frac{1}{2}\begin{pmatrix} x_1 - \mu_G \\ \ln x_2 - \mu_L \end{pmatrix}^T \boldsymbol{\Sigma}_{\mathrm{mx}}^{-1} \begin{pmatrix} x_1 - \mu_G \\ \ln x_2 - \mu_L \end{pmatrix}\right\}$$

Where

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$$\boldsymbol{\Sigma}_{mx} = \begin{pmatrix} VAR(X_1) & COV(X_1, \ln X_2) \\ COV(X_1, \ln X_2) & VAR(\ln X_2) \end{pmatrix}$$

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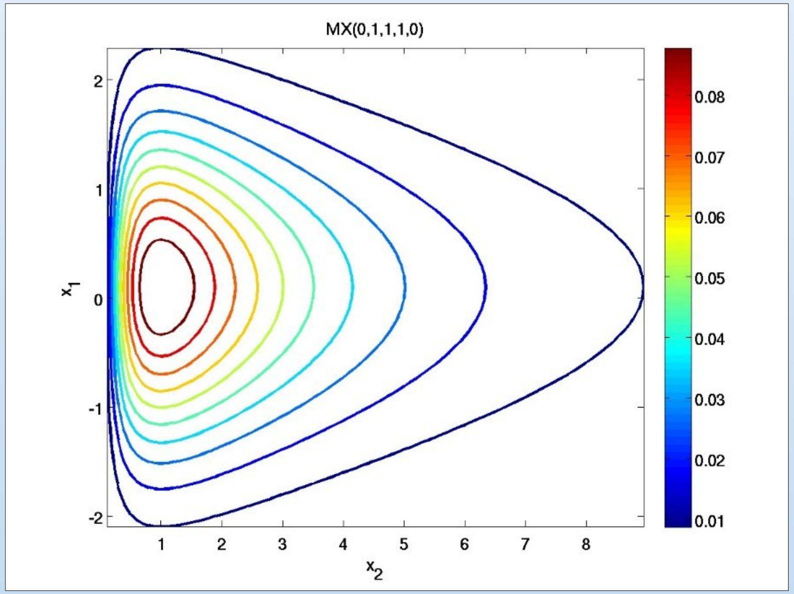
Properties of the Mixed Distribution

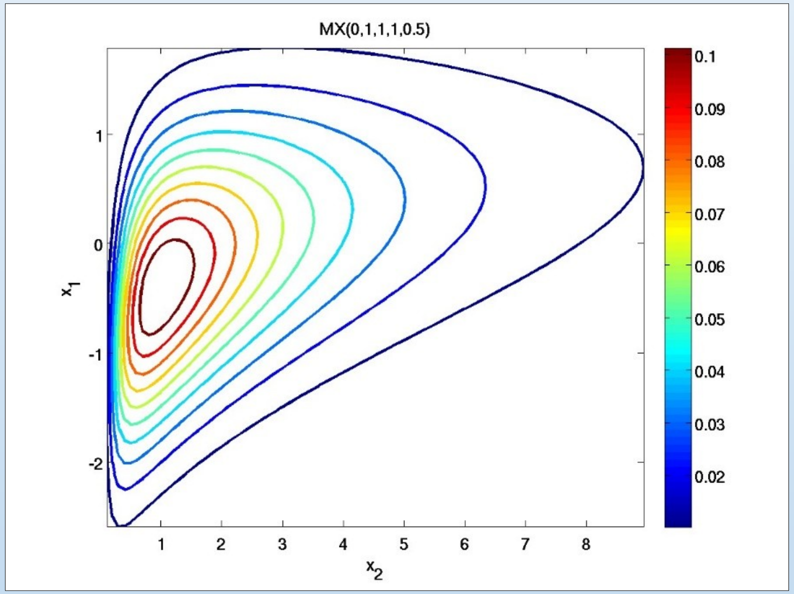
- An important property of the mixed distribution is the definitions of the three descriptive statistics. The mean for each component can be found through forming the marginal and joint pdfs which can be shown to be Gaussian and lognormal, or vice-versa. Therefore the mean, mode and median are given by

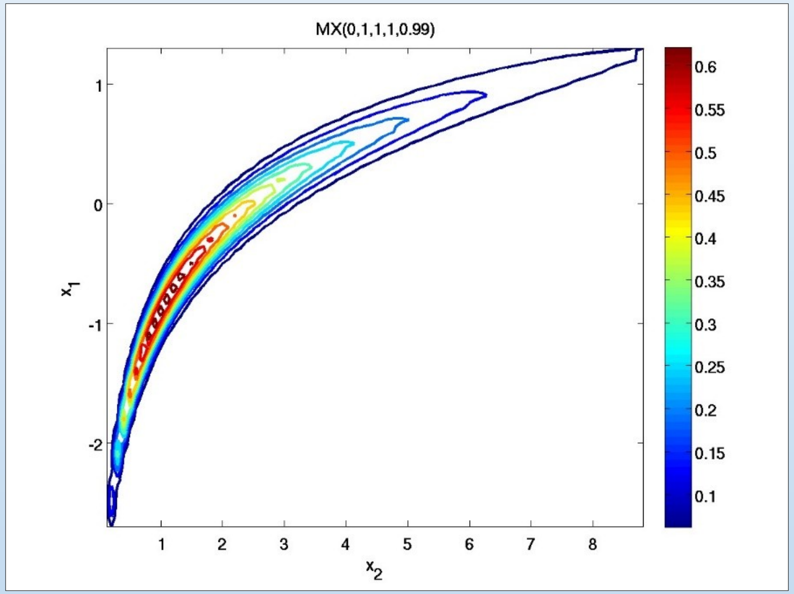
$$mean \equiv \begin{pmatrix} \mu_G \\ \exp\left\{\mu_L + \frac{\sigma_L^2}{2}\right\} \end{pmatrix}, \qquad median \equiv \begin{pmatrix} \mu_G \\ \exp\{\mu_L\} \end{pmatrix},$$

$$mode \equiv \begin{pmatrix} \mu_G - \rho\sigma_G\sigma_L \\ \exp\mu_L - \sigma_L^2 \end{pmatrix}$$

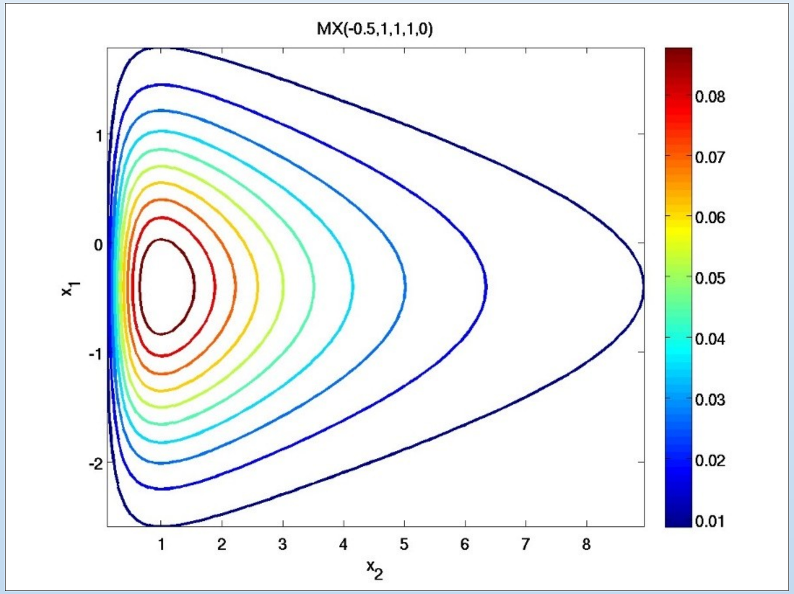
Plots of the Mixed Distribution

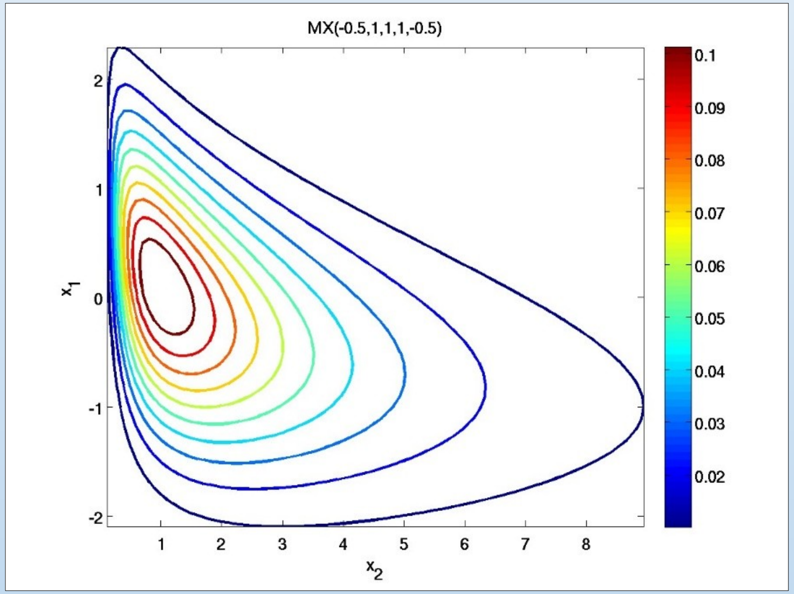


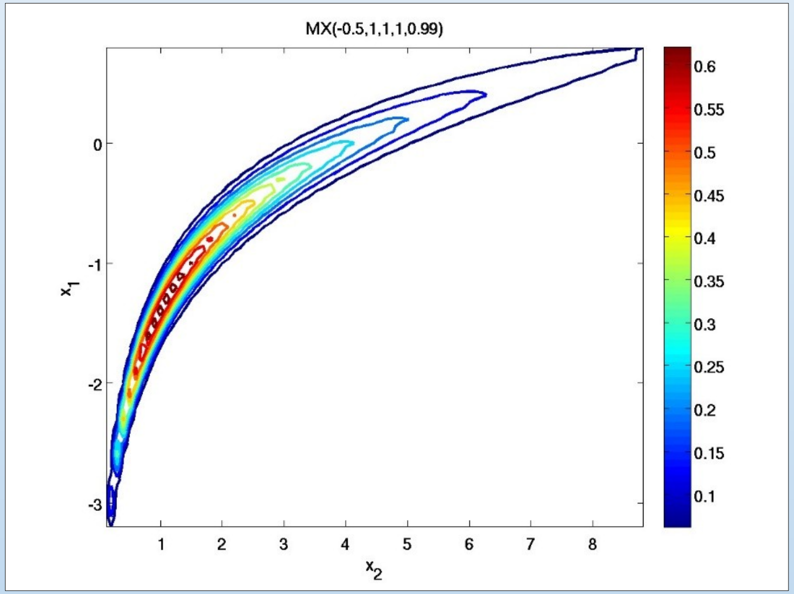




Plots of the Mixed Distribution







Applying the Mixed Distribution to VAR

- To be able to apply the mixed distribution to a variational formulation we require the definitions for the errors along with the multivariate version of the mixed distribution. The background and observational errors are given by

$$\boldsymbol{\varepsilon}_b \equiv \begin{pmatrix} \boldsymbol{x}_{p_1}^t - \boldsymbol{x}_{p_1}^b \\ \dfrac{\boldsymbol{x}_{q_1}^t}{\boldsymbol{x}_{q_1}^b} \end{pmatrix}, \qquad \boldsymbol{\varepsilon}_o \equiv \begin{pmatrix} \boldsymbol{y}_{p_2} - \boldsymbol{h}_{p_2}(\boldsymbol{x}) \\ \dfrac{\boldsymbol{y}_{q_2}}{\boldsymbol{h}_{q_2}(\boldsymbol{x})} \end{pmatrix}$$

Where there are different number of Gaussian and observational background and observational errors, and that $N = p_1 + q_1$ and $N_o = p_2 + q_2$.

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$$MX(\boldsymbol{\mu}_{mx}, \boldsymbol{\Sigma}_{\mathrm{mx}}) \equiv \frac{1}{\sqrt{|\boldsymbol{\Sigma}_{mx}|(2\pi)^{\frac{N}{2}}}} \prod_{i=p+1}^{N} \frac{1}{x_i} \exp\left\{-\frac{1}{2}\begin{pmatrix} \boldsymbol{x}_p - \boldsymbol{\mu}_p \\ \ln \boldsymbol{x}_2 - \boldsymbol{\mu}_q \end{pmatrix}^T \boldsymbol{\Sigma}_{\mathrm{mx}}^{-1} \begin{pmatrix} \boldsymbol{x}_p - \boldsymbol{\mu}_p \\ \ln \boldsymbol{x}_2 - \boldsymbol{\mu}_q \end{pmatrix}\right\}$$

Where

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$$\boldsymbol{\mu}_{\mathrm{mx}} \equiv \begin{pmatrix} \boldsymbol{\mu}_p \\ \boldsymbol{\mu}_q \end{pmatrix} \quad \boldsymbol{\Sigma}_{\mathrm{mx}} \equiv \begin{pmatrix} \boldsymbol{\Sigma}_{pp} & \boldsymbol{\Sigma}_{pq} \\ \boldsymbol{\Sigma}_{qp} & \boldsymbol{\Sigma}_{\mathrm{qq}} \end{pmatrix}$$

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$$\boldsymbol{x}_{\boldsymbol{mode}} = \begin{pmatrix} \boldsymbol{\mu}_p - \langle \boldsymbol{\Sigma}_{pq}, \mathbf{1}_q \rangle \\ \exp\{\boldsymbol{\mu}_q - \langle \boldsymbol{\Sigma}_{qq}, \mathbf{1}_q \rangle\} \end{pmatrix}$$

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$$
\begin{aligned}
J_{mx}(\boldsymbol{x}^t)
&= \frac{1}{2}\begin{pmatrix} \boldsymbol{x}_{p_1}^t - \boldsymbol{x}_{p_1}^b \\ \ln \boldsymbol{x}_{q_1}^t - \ln \boldsymbol{x}_{q_1}^b \end{pmatrix}^{\mathrm{T}} \boldsymbol{B}_{\mathrm{mx}}^{-1} \begin{pmatrix} \boldsymbol{x}_{p_1}^t - \boldsymbol{x}_{p_1}^b \\ \ln \boldsymbol{x}_{q_1}^t - \ln \boldsymbol{x}_{q_1}^b \end{pmatrix} + \left\langle \begin{pmatrix} \boldsymbol{x}_{p_1}^t - \boldsymbol{x}_{p_1}^b \\ \ln \boldsymbol{x}_{q_1}^t - \ln \boldsymbol{x}_{q_1}^b \end{pmatrix}, \begin{pmatrix} \boldsymbol{0}_{p_1} \\ \boldsymbol{1}_{q_1} \end{pmatrix} \right\rangle \\
&+ \frac{1}{2}\begin{pmatrix} \boldsymbol{y}_{p_2} - \boldsymbol{h}_{p_2}(\boldsymbol{x}^t) \\ \ln \boldsymbol{y}_{q_2} - \ln \boldsymbol{h}_{q_2}(\boldsymbol{x}^t) \end{pmatrix}^{T} \boldsymbol{R}_{mx}^{-1} \begin{pmatrix} \boldsymbol{y}_{p_2} - \boldsymbol{h}_{p_2}(\boldsymbol{x}^t) \\ \ln \boldsymbol{y}_{q_2} - \ln \boldsymbol{h}_{q_2}(\boldsymbol{x}^t) \end{pmatrix} \\
&+ \left\langle \begin{pmatrix} \boldsymbol{y}_{p_2} - \boldsymbol{h}_{p_2}(\boldsymbol{x}^t) \\ \ln \boldsymbol{y}_{q_2} - \ln \boldsymbol{h}_{q_2}(\boldsymbol{x}^t) \end{pmatrix}, \begin{pmatrix} \boldsymbol{0}_{p_2} \\ \boldsymbol{1}_{q_2} \end{pmatrix} \right\rangle
\end{aligned}
$$

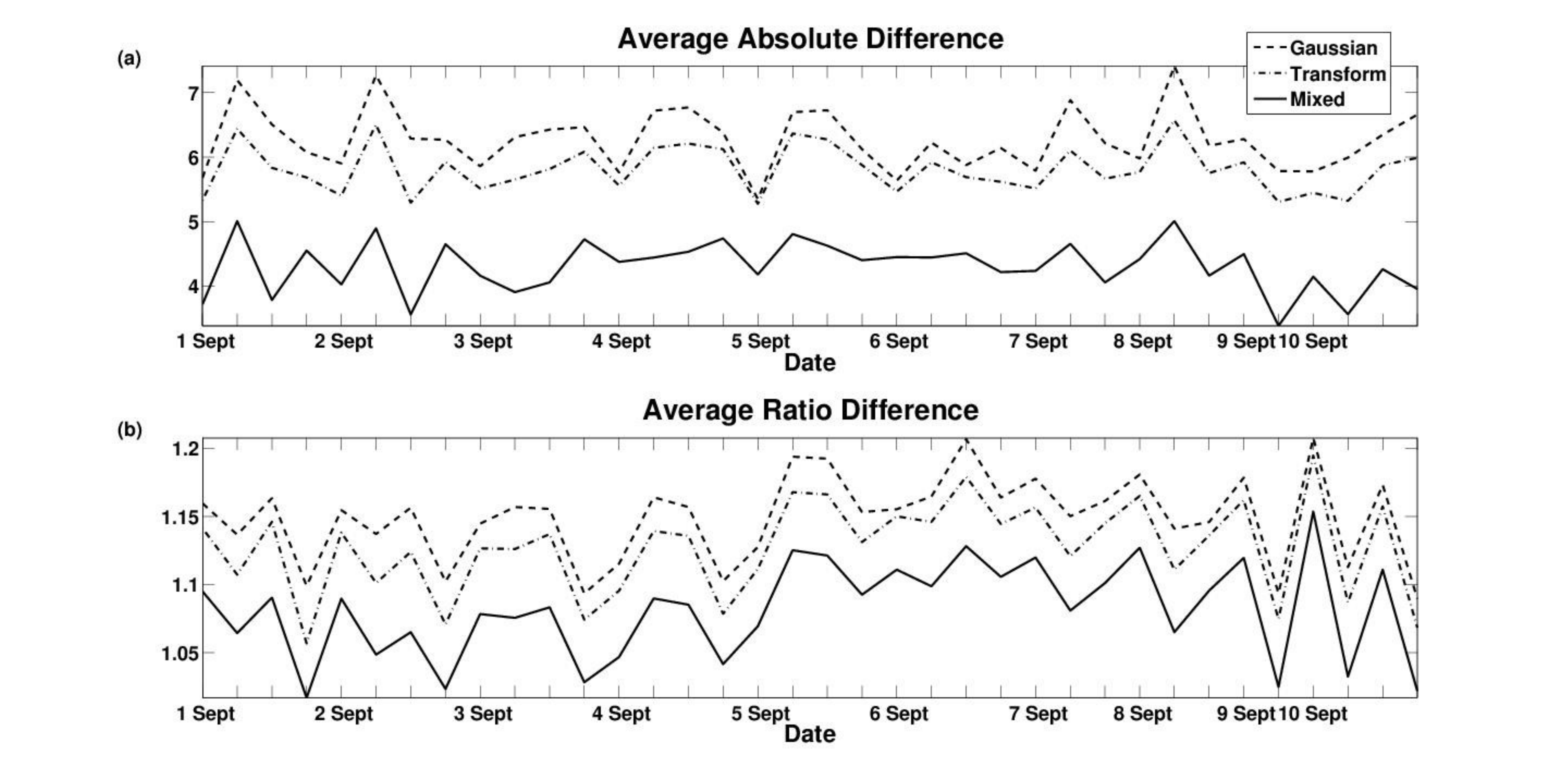
Application of the Mixed Distribution

The CIRA 1-Dimensional Optimal Estimator (C1DOE) is a 1DVAR retrieval system for mixing-ratio and temperature from microwave brightness temperatures. Its original version is a Gaussian fits all formulation.

We have now implemented the mixed distribution approach where we are assuming lognormal errors for the mixing-ratio, and Gaussian for the temperature.

Along with the mixed distribution and Gaussian fits all approaches we have also implemented the logarithmic transform approach for mixing-ratio (Kliewer et al 2016).

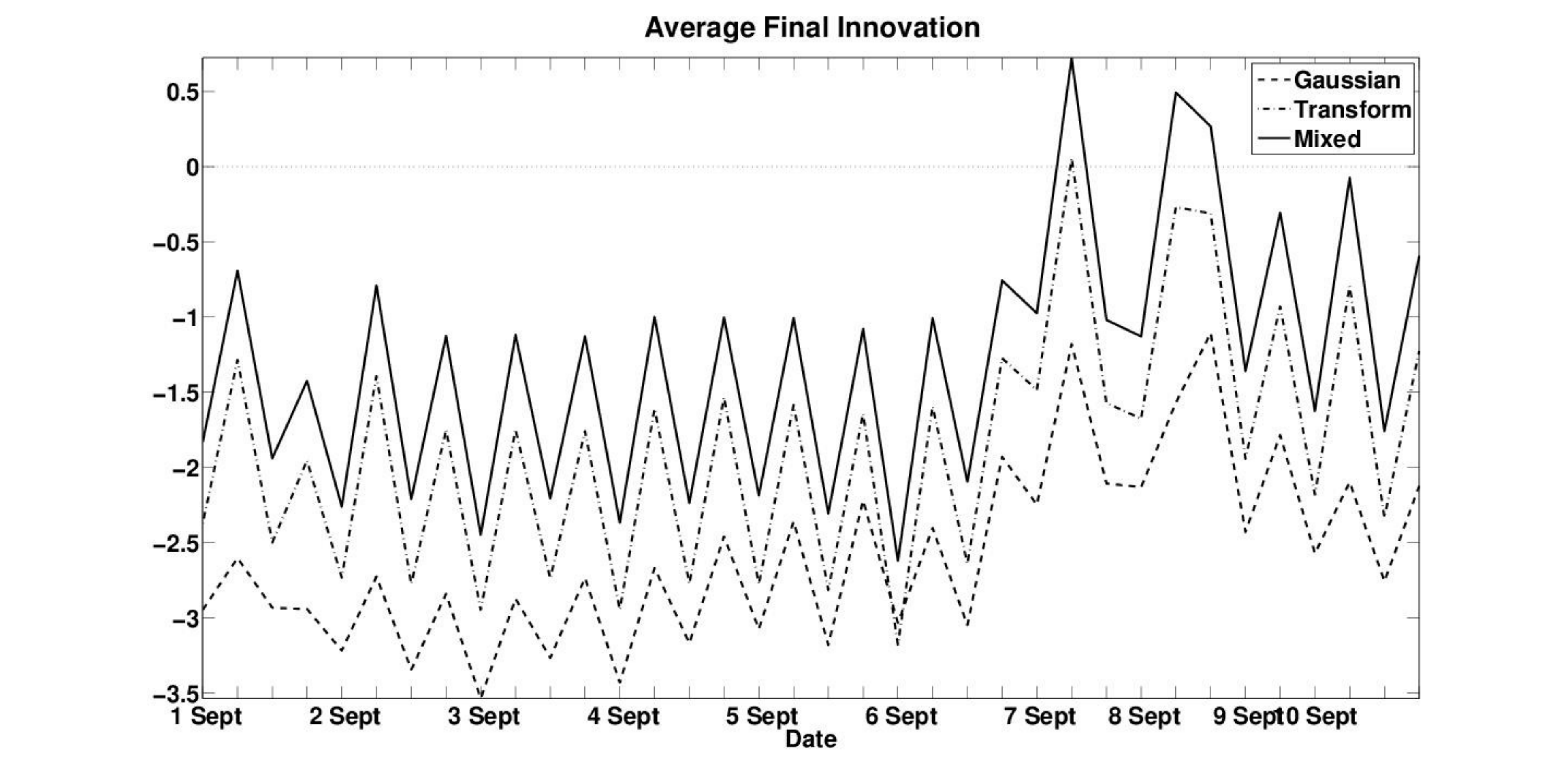
Application of the Mixed Distribution



Comparisons of the three retrieval methods against the Microwave Surface and Precipitation Products Systems (MSPPS) TPW product. Solid is the mixed approach, dot-dashed is the transform and the dashed is the Gaussian.

Application of the Mixed Distribution

AMSU-A Channel 6 (54.4GHz) (Temperature Channel in the troposphere) Final Innovations



Lognormal Based Quality Control Measures

- The gross observational error quality control measure is roughly given by

$$|y - h(x_b)| < 3\sigma_o,$$

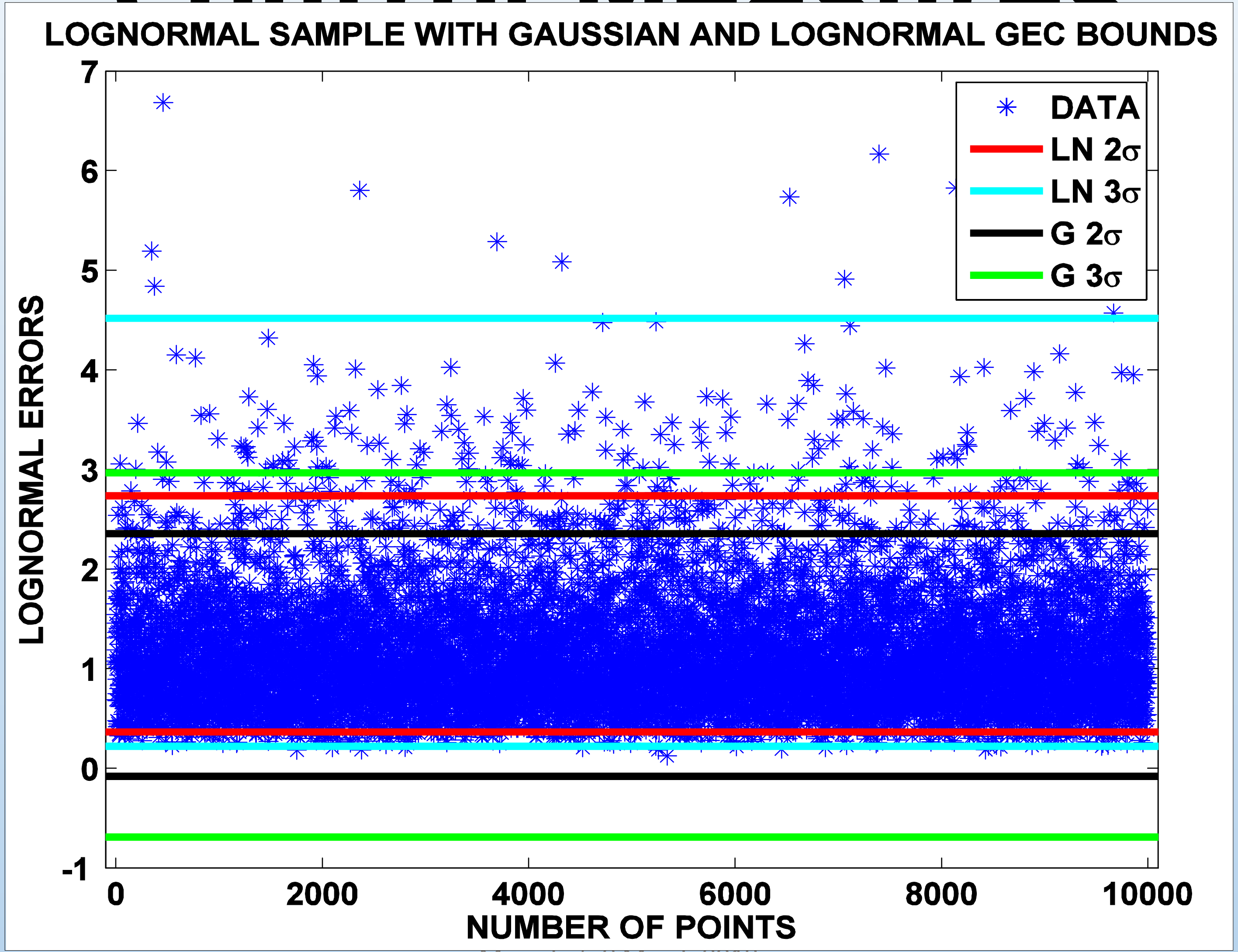
where σ_o is the observational error's standard deviation. This measure is accounting for approximately 99% of the distribution.

Given that this measure is consistent with percentiles, we would consider something similar for lognormally distributed errors.

If we apply the logarithmic transform to the lognormal random variable to make it a Gaussian random variable, then we could simply use the property of the **preservation of percentiles**, which means that equivalent lognormal gross error check could be

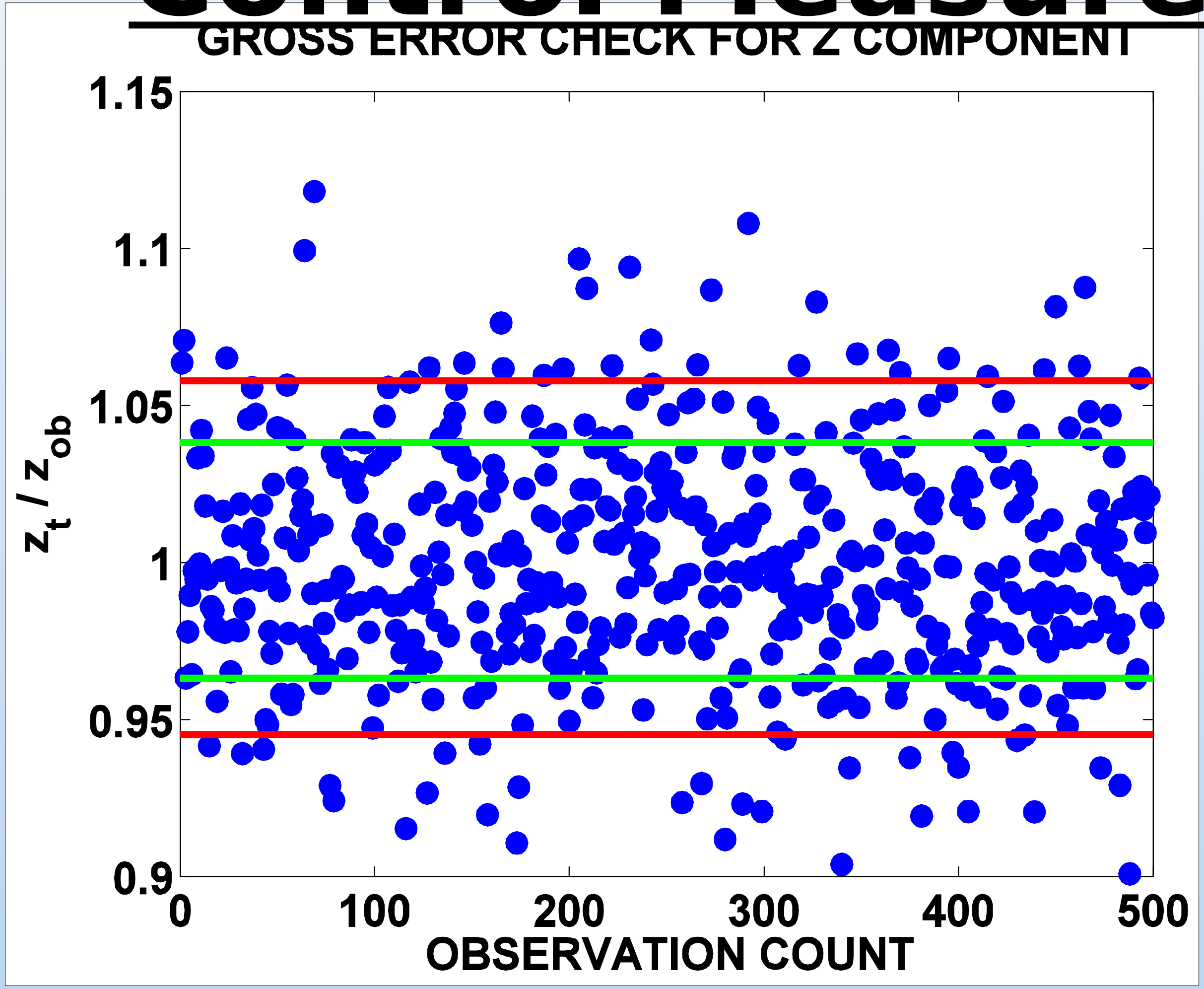
$$|\ln y - \ln h(x)| < 3\sigma_{o,L} \quad or \quad \exp\{-3\sigma_{o,L}\} < \frac{y}{h(x)} < \exp\{3\sigma_{o,L}\}.$$

Lognormal Based Quality Control Measures



Plot of a 10000 lognormal random numbers from $LN(0, 0.5)$ with the equivalent Gaussian and lognormal gross error check bounds.

Lognormal Based Quality Control Measures



An example from the Lorenz 63 model where the observations were generated from a lognormal random number generator multiplying the z_t with values for the mean and standard deviation of 0 and 0.0375 respectively. The green line is 2 SD, whilst the red is 3 SD.

Conclusions and Further Work

- Have presented a multivariate PDF that is a models the behavior of Gaussian and lognormal random variables simultaneously.
- Have shown that the Gaussian component of the mode of the distribution is a function of the covariances between the Gaussian and lognormal random variables.
- Presented results of applying the mixed distribution in a 1DVAR retrieval system, and compared its performance against Gaussian fits all and the logarithmic approach.
- Have presented some early ideas for the equivalent of the gross error check for lognormally distributed observation errors.

Conclusions and Further Work

- To develop the hybrid version of the mixed distribution based variational data assimilation
- Develop the buddy check and variational quality control measures
- Develop non-Gaussian detect algorithm to make the mixed distribution dynamically based.
- Implement the mixed distribution into the WRF-GSI for both the static and hybrid formulations.